

SATURATION AND ASSOCIATED PRIMES OF POWERS OF EDGE IDEALS

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ABSTRACT. For the edge ideal I of an arbitrary simple graph Γ we describe the monomials of the saturation of I^t in terms of (vertex) weighted graphs associated with the monomials. This description allows us to characterize the embedded associated primes of I^t as covers of Γ which contain certain types of subgraphs of Γ . As an application, we completely classify the associated primes of I^2 and I^3 in terms of Γ .

INTRODUCTION

Let Γ be a simple graph on the vertex set $V = \{1, 2, \dots, n\}$. The *edge ideal* of Γ is the ideal $I(\Gamma)$ generated by the monomials $x_i x_j$, $\{i, j\} \in \Gamma$, in the polynomial ring $R = k[x_1, \dots, x_n]$ over a field k . For simplicity we set $I = I(\Gamma)$.

The first goal of this paper is to describe the monomials of the saturation \tilde{I}^t of any power I^t , $t \geq 2$. The motivation comes from the facts that for every $\mathbf{a} \in \mathbb{N}^n$, the \mathbf{a} -component of the local cohomology modules of R/I^t can be computed by means of a simplicial complex $\Delta_{\mathbf{a}}$ and that the facets of $\Delta_{\mathbf{a}}$ can be described by the condition $x^{\mathbf{a}} \in \tilde{J}^t \setminus J^t$, where $x^{\mathbf{a}}$ denotes the monomial whose exponent vector is \mathbf{a} and J is the edge ideal of a subgraph of Γ (see Section 1 for more details). If we know the monomials of \tilde{J}^t we will be able to compute these local cohomology modules, which provides information on the depth and the Castelnuovo-Mumford regularity of R/I^t . In particular, the existence of a monomial in $\tilde{I}^t \setminus I^t$ is a criterion for the maximal homogeneous ideal to be an associated prime of I^t . Therefore, using localization we can obtain a combinatorial characterization of the associated primes of I^t , which is the second goal of this paper.

There have been several works on the behavior of the associated primes and the depth of I^t for t large enough (see e.g. [1], [2], [4], [5], [7], [9], [12], [16]), but little is known about a particular power I^t (except for its Cohen-Macaulay property [11], [14]). There was a combinatorial description of the associated primes of every power of the cover ideals of graphs by Francisco, Ha and Van Tuyl [3]. However, this result cannot be used to study edge ideals of graphs. Recently, Herzog and Hibi [6] gave a criterion for $\text{depth } R/I^2 = 0$ or, in other words, for the maximal homogeneous ideal to be an associated prime of I^2 in terms of Γ . A more general result was found independently by Terai and Trung [15] who gave a combinatorial characterization of the associated primes of the second power of an arbitrary squarefree monomial ideal. So it is a challenge to

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find a similar characterization of the associated primes of I^t for $t \geq 3$. The results of this paper provide a general approach to this problem.

Our idea is to represent a monomial $x^{\mathbf{a}}$ by a (vertex) weighted graph $\Gamma_{\mathbf{a}}$, which is obtained from Γ by assigning every vertex i with the weight a_i , where a_i is the exponent of x_i in $x^{\mathbf{a}}$. Our first result is a combinatorial criterion for $x^{\mathbf{a}} \in \tilde{I}^t \setminus I^t$ in terms of $\Gamma_{\mathbf{a}}$. More precisely, we show that $x^{\mathbf{a}} \in \tilde{I}^t \setminus I^t$ if and only if the matching numbers of $\Gamma_{\mathbf{a}}$ and of certain induced subgraphs of $\Gamma_{\mathbf{a}}$ satisfy some bounds (see Theorem 2.1). These bounds impose strong conditions on the set $V_{\mathbf{a}} := \{i \in V \mid a_i > 0\}$. For instance, we can show that every vertex of $V \setminus V_{\mathbf{a}}$ is adjacent to a vertex of $V_{\mathbf{a}}$ and that every connected component of the induced subgraph of Γ on $V_{\mathbf{a}}$ contains an odd cycle of length $\leq 2t - 1$. In particular, we can show that $\deg x^{\mathbf{a}} \leq 3(t - 1)$ if $x^{\mathbf{a}} \in \tilde{I}^t \setminus I^t$. This might have consequences on the Castelnuovo-Mumford regularity of R/I^t .

It is known that an associated prime of I^t is of the form P_F , where F is a cover of Γ and P_F denotes the ideal generated by the variables x_i , $i \in F$. In particular, the minimal associated primes of I^t correspond to the minimal covers of Γ . Using the description of the monomials in $\tilde{I}^t \setminus I^t$ and the technique of localization we are able to give a criterion for P_F to be an embedded (i.e. non-minimal) associated prime of I^t in terms of certain weighted graphs related to F (see Theorem 2.4). As a consequence, we obtain a simple necessary condition for P_F to be an embedded associated prime of I^t , namely that F is minimal among the covers of Γ containing the closed neighborhood of a set $U \subseteq V$ such that every connected components of Γ_U contains at least one odd cycle of length $\leq 2t - 1$.

In a similar manner, we can also give a sufficient condition for P_F to be an embedded associated prime of I^t , which depends only on the existence of a special weighted graph on Γ (see Theorem 3.3). It turns out that the above necessary condition is also sufficient for P_F to be an embedded associated prime of some power of I . Moreover, we can give an upper bound for the least number t_0 such that P_F is an associated prime of I^t for all $t \geq t_0$ (see Theorem 3.5). The proof is inspired by the technique of adding edges to an odd cycle by Chen, Morey and Sung [2], who used it to give a characterization of the stable set $\text{Ass}^\infty(I)$ and an upper bound for the stability index $\text{astab}(I)$ of the sets of associated primes of I^t . Our approach yields a simpler characterization of $\text{Ass}^\infty(I)$ and a better upper bound for $\text{astab}(I)$.

Finally, to demonstrate the efficiency of our approach we show that the afore mentioned results on the associated primes of I^2 of Herzog and Hibi [6] and of Terai and Trung [15] are immediate consequences of the above criterion. Furthermore, we give a complete classification of the associated primes of I^3 . According to this classification, P_F is an embedded associated prime of I^3 if and only if F is a minimal cover or minimal among the covers of Γ containing the closed neighborhood of a subgraph of the following forms: a triangle, a union of an edge and a triangle meeting at a vertex, a union of two non-adjacent triangles, a union of two triangles meeting at a vertex, a pentagon.

1. SATURATION OF MONOMIAL IDEALS

Let \mathfrak{m} be the maximal homogeneous ideal of $R = k[x_1, \dots, x_n]$. Given an ideal I of R , one calls the ideal $\tilde{I} := \bigcup_{m \geq 1} I : \mathfrak{m}^m$ the *saturation* of I . In this section we will explain why the condition $x^{\mathbf{a}} \in \tilde{I} \setminus I$ is important for the computation of the local

cohomology modules of a monomial ideal and for the characterization of associated primes of powers of edge ideals.

Let $I \neq 0$ be a monomial ideal. Then R/I is an \mathbb{N}^n -graded algebra. Hence the local cohomology modules $H_{\mathfrak{m}}^i(R/I)$ are \mathbb{Z}^n -graded. Takayama [13] showed that for every degree $\mathbf{a} \in \mathbb{Z}^n$, the \mathbf{a} -component of $H_{\mathfrak{m}}^i(R/I)$ can be expressed in terms of the reduced homology $\tilde{H}_j(\Delta_{\mathbf{a}}, k)$ of a simplicial complex $\Delta_{\mathbf{a}}$, which is defined as follows.

Let $V = \{1, \dots, n\}$ and $G_{\mathbf{a}} := \{i \in V \mid a_i < 0\}$. For every subset $G \subseteq V$ let \mathbf{a}_G denote the vector obtained from \mathbf{a} by setting $a_i = 0$ for $i \in G$ and define

$$I_G := k[x_i \mid i \in V \setminus G] \cap IR[x_i^{-1} \mid i \in G].$$

Then the set of the facets of $\Delta_{\mathbf{a}}$ is given by the formula

$$\mathcal{F}(\Delta_{\mathbf{a}}) = \{G \setminus G_{\mathbf{a}} \mid G_{\mathbf{a}} \subseteq G \subseteq V, x^{\mathbf{a}_G} \in \widetilde{I_G} \setminus I_G\}.$$

This description of $\Delta_{\mathbf{a}}$ is taken from [15, Lemma 1.3], which is simpler than the original definition in [13].

Let Δ be the simplicial complex on V such that \sqrt{I} is the Stanley-Reisner ideal of Δ . This means \sqrt{I} is generated by the monomials $x_{i_1} \cdots x_{i_r}$, $\{i_1, \dots, i_r\} \notin \Delta$. For $j = 1, \dots, n$, we denote by ρ_j the maximum of positive j -th coordinates of all vectors $\mathbf{b} \in \mathbb{N}^n$ such that $x^{\mathbf{b}}$ is a minimal generator of I . Then the result of Takayama can be formulated as follows.

Theorem 1.1. [13, Theorem 1]

$$\dim_k H_{\mathfrak{m}}^i(R/I)_{\mathbf{a}} = \begin{cases} \dim_k \tilde{H}_{i-|G_{\mathbf{a}}|-1}(\Delta_{\mathbf{a}}, k) & \text{if } G_{\mathbf{a}} \in \Delta \text{ and} \\ & a_j < \rho_j \text{ for } j = 1, \dots, n, \\ 0 & \text{else.} \end{cases}$$

To compute the facets of $\Delta_{\mathbf{a}}$ we have to check the condition $x^{\mathbf{a}_G} \in \widetilde{I_G} \setminus I_G$. In particular, this condition implies $\widetilde{I_G} \neq I_G$. We shall see that a set G with the property $\widetilde{I_G} \neq I_G$ corresponds to an associated prime of I . For a subset F of V , we denote by P_F the ideal of R generated by the variables x_i , $i \in F$. Then every associated prime of I has the form P_F .

Lemma 1.2. $\widetilde{I_G} \neq I_G$ if and only if $P_{V \setminus G}$ is an associated prime of I .

Proof. Set $A := k[x_i \mid i \in V \setminus G]$ and let Q be the maximal homogeneous ideal of A . Then $\widetilde{I_G} \neq I_G$ if and only if Q is an associated prime of I_G . Since $P_{V \setminus G} = QR$ and $R = A[x_i \mid i \in G]$ is a polynomial ring over A , Q is an associated prime of I_G if and only if $P_{V \setminus G}$ is an associated prime of $I_G R$. Set $B := R[x_i^{-1} \mid i \in G]$. Since B is a localization of R and $P_{V \setminus G} \neq B$, $P_{V \setminus G}$ is an associated prime of $I_G R$ if and only if $P_{V \setminus G} B$ is an associated prime of $I_G B$. By definition, I_G is generated by the monomials obtained from the monomials of I by removing the variables x_i , $i \in G$. Thus, every monomial of I is divisible by a monomial of I_G . Hence $IB \subseteq I_G B$. On the other hand, $I_G B \subseteq IB$ because $I_G = A \cap IB$. Therefore, $I_G B = IB$. Since $P_{V \setminus G} B$ is an associated prime of IB if and only if $P_{V \setminus G}$ is an associated prime of I , we get the conclusion. \square

By Lemma 1.2, we only need to check the condition $x^{\mathbf{a}_G} \in \widetilde{I_G} \setminus I_G$ for G with the property that $P_{V \setminus G}$ is an associated prime of I . Therefore, the associated primes of I play an important role in the computation of the local cohomology modules of R/I .

From now on let I be the edge ideal of a simple graph Γ on the vertex set V . We shall see that the description of the associated primes of a power I^t can be reduced to the problem when \mathfrak{m} is an associated prime of I^t . It is well known that P_F is a minimal associated prime of I^t if and only if F is a minimal (vertex) cover of G . So we have to find the associated primes of I^t among the ideals P_F , where F is a cover of Γ .

Let $\text{core}(F)$ denote the set of vertices in F which are not adjacent to any vertex in $V \setminus F$. Note that a cover F is minimal if and only if $\text{core}(F) = \emptyset$. Let Γ_U denote the induced subgraph of Γ on subset U of V . In the following we set $I(\Gamma_{\text{core}(F)}) = 0$ if $\text{core}(F) = \emptyset$.

Proposition 1.3. *Let F be a cover of Γ . Let $J = I(\Gamma_{\text{core}(F)})$. Let \mathfrak{n} denote the maximal homogeneous ideal of $k[x_i \mid i \in \text{core}(F)]$. Then P_F is an associated prime of I^t if and only if \mathfrak{n} is an associated prime of J^t .*

Proof. Set $A := k[x_i \mid i \in F]$ and let Q be the maximal homogeneous ideal of A . By the proof of Lemma 1.2, P_F is an associated prime of I^t if and only if Q is an associated prime of $(I^t)_G$, where $G = V \setminus F$. By definition, I_G and $(I^t)_G$ are generated by the monomials obtained from the monomials of I and I^t by removing the variables x_i , $i \notin F$. From this it follows that $(I^t)_G = (I_G)^t$. For all $j \in F \setminus \text{core}(F)$, there exists $i \notin F$ adjacent to j . Since $x_i x_j \in I$, we get $x_j \in I_G$. The monomials of I_G which do not contain any variable x_j , $j \in F \setminus \text{core}(F)$, belong to J . Since $J \subset I_G$, this implies

$$I_G = (x_j \mid j \in F \setminus \text{core}(F))A + JA.$$

Note that the ideals $(x_j \mid j \notin F \setminus \text{core}(F))$ and J are generated by monomials in two disjoint sets of variables. Then Q is an associated prime of $(I_G)^t$ if and only if \mathfrak{n} is an associated prime of J^s for some $s \leq t$ [2, Lemma 2.1]. By [9, Theorem 2.15], the latter condition implies that \mathfrak{n} is also an associated prime of J^t . Therefore, P_F is an associated prime of I^t if and only if \mathfrak{n} is an associated prime of J^t . \square

By Lemma 1.3, the description of the associated primes of I^t can be reduced to problem when there does exist a monomial $x^{\mathbf{a}} \in \widetilde{J^t} \setminus J^t$ because this condition is a criterion for \mathfrak{n} to be an associated prime of J^t . Furthermore, the condition $x^{\mathbf{a}G} \in \widetilde{(I^t)_G} \setminus (I^t)_G$, which appears in the definition of the complex $\Delta_{\mathbf{a}}$ of I^t , is in fact equivalent to the condition $x^{\mathbf{a}} \in \widetilde{J^s} \setminus J^s$ for some edge ideal J and some $s \leq t$.

Proposition 1.4. *Let $F = V \setminus G$ and $J = I(\Gamma_{\text{core}(F)})$. Then $x^{\mathbf{a}G} \in \widetilde{(I^t)_G} \setminus (I^t)_G$ if and only if $x^{\mathbf{a}_{V \setminus \text{core}(F)}} \in \widetilde{J^s} \setminus J^s$, where $s = t - \sum_{i \in F \setminus \text{core}(F)} a_i$.*

Proof. By the proof of Proposition 1.3 we have $(I^t)_G = (I_G)^t$ and

$$I_G = (x_j \mid j \in F \setminus \text{core}(F))A + JA.$$

This formula implies the following relations:

- (1) $(I_G)^t : x_j = (I_G)^{t-1}$ for all $j \in F \setminus \text{core}(F)$,
- (2) $(I_G)^t \cap S = J^t$, where $S = k[x_i \mid i \in \text{core}(F)]$.

Note that $x^{\mathbf{a}G} = x^{\mathbf{a}_{V \setminus \text{core}(F)}} \prod_{j \in F \setminus \text{core}(F)} x_j^{a_j}$. Then $x^{\mathbf{a}G} \in (I_G)^t$ if and only if $x^{\mathbf{a}_{V \setminus \text{core}(F)}} \in (I_G)^t : \prod_{j \in F \setminus \text{core}(F)} x_j^{a_j} = (I_G)^s$, where the last equality follows from (1). By definition, $x^{\mathbf{a}_{V \setminus \text{core}(F)}} \in S$. It follows from (2) that $x^{\mathbf{a}_{V \setminus \text{core}(F)}} \in (I_G)^s$ if and only if $x^{\mathbf{a}_{V \setminus \text{core}(F)}} \in J^s$. Therefore, $x^{\mathbf{a}G} \in (I_G)^t$ if and only if $x^{\mathbf{a}_{V \setminus \text{core}(F)}} \in J^s$.

Similarly, we can show that for $i \in \text{core}(F)$, $x^{\mathbf{a}_G} \in \bigcup_{m \geq 1} (I_G)^t : x_i^m$ if and only if $x^{\mathbf{a}_{V \setminus \text{core}(F)}} \in \bigcup_{m \geq 1} J^s : x_i^m$. Since $\bigcup_{m \geq 1} (I_G)^t : x_i^m = A$ for $i \in F \setminus \text{core}(F)$, we have

$$\widetilde{(I_G)^t} = \bigcup_{m \geq 1} (I_G)^t : Q^m = \bigcup_{m \geq 1} \bigcap_{i \in F} (I_G)^t : x_i^m = \bigcup_{m \geq 1} \bigcap_{i \in \text{core}(F)} (I_G)^t : x_i^m.$$

Therefore, $x^{\mathbf{a}_G} \in \widetilde{(I_G)^t}$ if and only if

$$x^{\mathbf{a}_{V \setminus \text{core}(F)}} \in \bigcup_{m \geq 1} \bigcap_{i \in \text{core}(F)} J^s : x_i^m = \bigcup_{m \geq 1} J^s : \mathbf{n}^m = \widetilde{J^s}.$$

So we can conclude that $x^{\mathbf{a}_G} \in \widetilde{(I_G)^t} \setminus (I_G)^t$ if and only if $x^{\mathbf{a}_{V \setminus \text{core}(F)}} \in \widetilde{J^s} \setminus J^s$. \square

2. SATURATING WEIGHTED GRAPHS

Let I be the edge ideal of a simple graph Γ on the vertex set $V = \{1, \dots, n\}$. The aim of this section is to characterize the condition $x^{\mathbf{a}} \in \widetilde{I^t} \setminus I^t$ for a given vector $\mathbf{a} \in \mathbb{N}^n$ and to describe the associated primes of I^t in terms of Γ .

We shall need the notion of (vertex) weighted graphs. A *weighted graph* is a simple graph whose vertices are assigned with positive integers called *weight*. The simple graph is called the *base graph* of the weighted graph. We always consider a simple graph as a weighted graph whose vertices have the trivial weight 1.

Let Ω be an arbitrary weighted graph on a vertex set U . A *matching* of Ω is a family of edges (not necessarily different) such that every vertex of Ω appears in these edges no more times than its weight. The maximal number of edges of the matchings of Ω is called the *matching number*, denoted by $\nu(\Omega)$. For every subset N of U we denote by $\Omega - N$ the induced weighted subgraph of Ω on the vertex set $U \setminus N$.

Let $V_{\mathbf{a}} := \{i \in V \mid a_i > 0\}$. If we assign each vertex $i \in V_{\mathbf{a}}$ with the weight a_i , we obtain a weighted graph called $\Gamma_{\mathbf{a}}$. For a vertex $i \in V$, we denote by $N_{\mathbf{a}}(i)$ the set of all adjacent vertices of i in $\Gamma_{\mathbf{a}}$ and set $\deg_{\mathbf{a}}(i) = \sum_{j \in N_{\mathbf{a}}(i)} a_j$. Using these notions we can translate the condition $x^{\mathbf{a}} \in \widetilde{I^t} \setminus I^t$ in combinatorial terms as follows.

Theorem 2.1. $x^{\mathbf{a}} \in \widetilde{I^t} \setminus I^t$ if and only if the following conditions are satisfied:

- (i) $\nu(\Gamma_{\mathbf{a}}) < t$,
- (ii) $\nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)) \geq t - \deg_{\mathbf{a}}(i)$ for all $i \in V$.

Proof. First we will show that $x^{\mathbf{a}} \notin I^t$ if and only if condition (i) is satisfied. Let \mathbf{e}_i be the i -th unit vector of \mathbb{N}^n . It is clear that $x^{\mathbf{a}} \in I^t$ if and only if there exists a family of t edges $\{i_1, j_1\}, \dots, \{i_t, j_t\}$ of Γ (not necessarily different) such that $x^{\mathbf{a}}$ is divisible by the product $(x_{i_1} x_{j_1}) \cdots (x_{i_t} x_{j_t})$. The divisibility implies that every vertex of $V_{\mathbf{a}}$ appears in these edges no more times than its weight. Hence these edges form a matching of $\Gamma_{\mathbf{a}}$. Thus, $x^{\mathbf{a}} \in I^t$ if and only if $\nu(\Gamma_{\mathbf{a}}) \geq t$.

It remains to show that $x^{\mathbf{a}} \in \widetilde{I^t}$ if and only if condition (ii) is satisfied. By definition, $x^{\mathbf{a}} \in \widetilde{I^t}$ if and only if $x_i^m x^{\mathbf{a}} \in I^t$ for all $i \in V$ and $m \gg 0$. As we have seen above, this is equivalent to the condition $\nu(\Gamma_{\mathbf{a} + m\mathbf{e}_i}) \geq t$. Let Ω be the weighted subgraph of $\Gamma_{\mathbf{a} + m\mathbf{e}_i}$ whose edges contain at least one vertex in $N_{\mathbf{a}}(i)$ and whose vertices have the same weight as in $\Gamma_{\mathbf{a} + m\mathbf{e}_i}$. Then the number of edges of a matching of Ω can not exceed the appearing times of the vertices of $N_{\mathbf{a}}(i)$ in these edges. Hence $\nu(\Omega) \leq \sum_{j \in N_{\mathbf{a}}(i)} a_j = \deg_{\mathbf{a}}(i)$. If $m \geq \deg_{\mathbf{a}}(i)$, there is a matching of Ω which consists of $\deg_{\mathbf{a}}(i)$ edges

connecting i with every vertex $j \in N_{\mathbf{a}}(i)$ a_j times. Thus, $\nu(\Omega) = \deg_{\mathbf{a}}(i)$. Since the union of this matching of Ω with any matching of $\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)$ is a matching of $\Gamma_{\mathbf{a}+m\mathbf{e}_i}$,

$$\nu(\Gamma_{\mathbf{a}+m\mathbf{e}_i}) \geq \deg_{\mathbf{a}}(i) + \nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i))$$

for $m \gg 0$. On the other hand, since every matching of $\Gamma_{\mathbf{a}+m\mathbf{e}_i}$ is the disjoint union of a matching of Ω and a matching of $\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)$,

$$\nu(\Gamma_{\mathbf{a}+m\mathbf{e}_i}) \leq \nu(\Omega) + \nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)) = \deg_{\mathbf{a}}(i) + \nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)).$$

Therefore, $\nu(\Gamma_{\mathbf{a}+m\mathbf{e}_i}) = \deg_{\mathbf{a}}(i) + \nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i))$ for $m \gg 0$. So we can conclude that $x^{\mathbf{a}} \in \tilde{I}^t$ if and only if $\deg_{\Gamma_{\mathbf{a}}}(i) + \nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)) \geq t$. \square

In graph theory a subset W of V (or a subgraph of Γ on W) is called *dominating* if every vertex of $V \setminus W$ is adjacent to at least one vertex of W .

Corollary 2.2. *If $x^{\mathbf{a}} \in \tilde{I}^t \setminus I^t$, then $V_{\mathbf{a}}$ is a dominating set of Γ .*

Proof. By Theorem 2.1 we have

$$\deg_{\mathbf{a}}(i) \geq t - \nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)) \geq t - \nu(\Gamma_{\mathbf{a}}) \geq t - (t - 1) = 1$$

for all $i \in V \setminus V_{\mathbf{a}}$. This implies that i is adjacent to at least one vertex in $V_{\mathbf{a}}$. \square

Example 2.3. Let Γ be a graph which contains a dominating cycle C of length $2t - 1$, $t \geq 2$. Let $\mathbf{a} \in \{0, 1\}^n$ such that $V_{\mathbf{a}}$ is the vertex set of C . Then $x^{\mathbf{a}} \in \tilde{I}^t \setminus I^t$. To see this we will show that the conditions (i) and (ii) of Theorem 2.1 are satisfied. First, we observe that $\Gamma_{\mathbf{a}}$ is the induced subgraph of Γ on $V_{\mathbf{a}}$. Since $|V_{\mathbf{a}}| = 2t - 1$, we must have $\nu(\Gamma_{\mathbf{a}}) < t$. Since $V_{\mathbf{a}}$ is a dominating subset of V , every vertex $i \in V \setminus V_{\mathbf{a}}$ is adjacent to some vertex of $V_{\mathbf{a}}$. This property also holds for $i \in V_{\mathbf{a}}$ because every vertex of $V_{\mathbf{a}}$ is adjacent to two vertices of C . Therefore, $N_{\mathbf{a}}(i)$ is a non-empty subset of $V_{\mathbf{a}}$ for all $i \in V$. Since $|N_{\mathbf{a}}(i)| = \deg_{\mathbf{a}}(i)$, we have $\deg_{\mathbf{a}}(i) > 0$. Since every vertex of $N_{\mathbf{a}}(i)$ is contained in two edges of C , the subgraph $\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)$ contain at least $2t - 1 - 2\deg_{\mathbf{a}}(i)$ edges of C . Since these edges does not form the cycle C , there are at least $t - \deg_{\mathbf{a}}(i)$ disjoint edges among them. Therefore, $\nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)) \geq t - \deg_{\mathbf{a}}(i)$ for all $i \in V$.

We note that condition (ii) of Theorem 1.3 may involve vertices outside of $\Gamma_{\mathbf{a}}$. To investigate the conditions involving only the vertices of $\Gamma_{\mathbf{a}}$ we call a weighted graph Ω on a vertex set U *t-saturating* if $\nu(\Omega) < t$ and $\nu(\Omega - N_{\Omega}(i)) \geq t - \deg_{\Omega}(i)$ for all $i \in U$, where $N_{\Omega}(i)$ denotes the set of all adjacent vertices of i and $\deg_{\Omega}(i)$ is the sum of the weights of the vertices of $N_{\Omega}(i)$.

Since $N_{\Gamma_{\mathbf{a}}}(i) = N_{\mathbf{a}}(i)$ and $\deg_{\Gamma_{\mathbf{a}}}(i) = \deg_{\mathbf{a}}(i)$ for $i \in V_{\mathbf{a}}$, Theorem 2.1 can be reformulated as $x^{\mathbf{a}} \in \tilde{I}^t \setminus I^t$ if and only if $\Gamma_{\mathbf{a}}$ is *t-saturating* and $\nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)) \geq t - \deg_{\mathbf{a}}(i)$ for all $i \in V \setminus V_{\mathbf{a}}$.

It is well known that \mathbf{m} is an associated prime of I^t if and only if there exists a monomial $x^{\mathbf{a}} \in \tilde{I}^t \setminus I^t$. Therefore, one can reformulate Theorem 2.1 as a criterion for \mathbf{m} to be an associated prime of I^t . In the following we will deduce from this criterion a combinatorial characterization of the embedded associated primes of I^t . Note that an embedded prime of I^t is of the form P_F , where F is a cover of Γ with $\text{core}(F) \neq \emptyset$.

For a subset U of V we denote by $N(U)$ the set of the vertices adjacent to some vertex of U and by $N[U]$ the union of U and $N(U)$. These sets are called the *open neighborhood* and the *closed neighborhood* of U in Γ .

Theorem 2.4. *Let F be a cover of Γ with $\text{core}(F) \neq \emptyset$. Then P_F is an embedded associated prime ideal of I^t if and only if F is minimal among the covers containing $N[V_{\mathbf{a}}]$ for some $\mathbf{a} \in \mathbb{N}^n$ such that $\Gamma_{\mathbf{a}}$ is t -saturating and $\nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)) \geq t - \deg_{\mathbf{a}}(i)$ for all $i \in \text{core}(F) \setminus V_{\mathbf{a}}$.*

Proof. By Proposition 1.3, P_F is an associated prime ideal of I^t if and only if $\tilde{J}^t \neq J^t$, where $J = I(\Gamma_{\text{core}(F)})$. The latter condition means that there exists $\mathbf{a} \in \mathbb{N}^n$ with $V_{\mathbf{a}} \subseteq \text{core}(F)$ such that $x^{\mathbf{a}} \in \tilde{J}^t \setminus J^t$. Obviously, $(\Gamma_{\text{core}(F)})_{V_{\mathbf{a}}} = \Gamma_{V_{\mathbf{a}}}$ and $(\Gamma_{\text{core}(F)})_{\mathbf{a}} = \Gamma_{\mathbf{a}}$. By Theorem 2.1, $x^{\mathbf{a}} \in \tilde{J}^t \setminus J^t$ if and only if $\Gamma_{\mathbf{a}}$ is t -saturating and $\nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)) \geq t - \deg_{\mathbf{a}}(i)$ for all $i \in \text{core}(F) \setminus V_{\mathbf{a}}$. It remains to show that the condition $V_{\mathbf{a}} \subseteq \text{core}(F)$ can be replaced by the condition F is minimal among the covers containing $N[V_{\mathbf{a}}]$.

If $V_{\mathbf{a}} \subseteq \text{core}(F)$, then $N[V_{\mathbf{a}}] \subseteq F$ by the definition of $\text{core}(F)$. As we have seen above, we may assume further assume that $x^{\mathbf{a}} \in \tilde{J}^t \setminus J^t$. Then $V_{\mathbf{a}}$ is a dominating set of $\Gamma_{\text{core}(F)}$ by Corollary 2.2. Hence $\text{core}(F) \subseteq N[V_{\mathbf{a}}]$. Thus, every vertex of $F \setminus N[V_{\mathbf{a}}]$ is adjacent to a vertex of $V \setminus F$. From this it follows that every set $F \setminus i$, $i \in F \setminus N[V_{\mathbf{a}}]$, is not a cover of Γ . Therefore, F is minimal among the covers containing $N[V_{\mathbf{a}}]$. Conversely, if F is minimal among the covers containing $N[V_{\mathbf{a}}]$, then $N[V_{\mathbf{a}}] \subseteq F$. Hence every vertex of $V_{\mathbf{a}}$ is not adjacent to any vertex of $V \setminus F$. This implies $V_{\mathbf{a}} \subseteq \text{core}(F)$. \square

By Theorem 2.4, to find the embedded associated primes of I^t we have first to find the subsets $U \subseteq V$ such that there exists a t -saturating weighted graph on U . In the following we establish properties of t -saturating weighted graphs which can be used to detect them.

In the following we denote by Ω a weighted graph on the vertex set U and by a_i the weight of a vertex $i \in U$. Recall that a vertex of is called a *leaf vertex* if it is adjacent to only a vertex of the graph.

Lemma 2.5. *Let Ω be a t -saturating weighted graph. Then*

- (i) $a_i < \min\{\deg_{\Omega}(i), \nu(\Omega) + 1\}$,
- (ii) $a_i \geq 2$ if i adjacent to a leaf vertex of the base graph of Ω .

Proof. One can find a family of $\min\{a_i, \deg_{\Omega}(i)\}$ edges of Ω containing i such that the appearing times of every vertex $j \in N_{\Omega}(i)$ does not exceed a_j . These edges can be added to an arbitrary matching of $\Omega - N_{\Omega}(i)$ to form a matching of Ω . Therefore,

$$\min\{a_i, \deg_{\Omega}(i)\} + \nu(\Omega - N_{\Omega}(i)) \leq \nu(\Omega).$$

By the definition of t -saturating weighted graph, $\deg_{\Omega}(i) + \nu(\Omega - N_{\Omega}(i)) \geq t > \nu(\Omega)$. Therefore, we must have $a_i < \deg_{\Omega}(i)$ and $a_i + \nu(\Omega - N_{\Omega}(i)) \leq \nu(\Omega)$. The latter inequality implies $a_i < \nu(\Omega) + 1$. So we obtain (i). If i is adjacent to a leaf vertex j , then $a_i = \deg_{\Omega}(j)$. As shown above, $\deg_{\Omega}(j) > a_j$. Since $a_j \geq 1$, this implies (ii). \square

For the proof of the next result we need to extend some notations on simple graphs to weighted graphs. Let M be a matching of a weighted graph Ω . For every vertex i let w_i be the appearing times of i in the edges of M . We say that i is an *unmatched vertex* of M if $w_i < a_i$. An *augmenting walk* of M is a sequence of vertices and edges, where each edge's endpoints are the preceding and following vertices in the sequence, which satisfy the following conditions:

- (i) The first and last vertices are unmatched vertices,
- (ii) The appearing time of every vertex i does not exceed a_i ,

(iii) The number of edges is odd and the family of the even edges is contained in M .

With these notations we can easily extend Berge's theorem (see e.g. [8, Theorem 1.2.1]) to weighted graph, which says that $|M| = \nu(\Omega)$ if and only if M does not have any augmenting walk. We leave the reader to check this fact.

Proposition 2.6. *Let Ω be a t -saturating weighted graph. Then $\sum_{i \in U} a_i \leq 3(t-1)$.*

Proof. Let Γ be the base graph of Ω and \mathbf{a} the weight vector of Ω . Then $\Gamma_{\mathbf{a}} = \Omega$ and $N_{\mathbf{a}}(i) = N_{\Omega}(i)$ and $\deg_{\mathbf{a}}(i) = \deg_{\Omega}(i)$ for every $i \in U$. Choose a number $m \geq \deg_{\mathbf{a}}(i)$. We have seen in the proof of Theorem 2.1 that $\nu(\Gamma_{\mathbf{a}+m\mathbf{e}_i}) = \deg_{\mathbf{a}}(i) + \nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i))$. Since $\Gamma_{\mathbf{a}}$ is t -saturating, $\deg_{\mathbf{a}}(i) + \nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)) \geq t > \nu(\Gamma_{\mathbf{a}})$. Therefore, $\nu(\Gamma_{\mathbf{a}+m\mathbf{e}_i}) > \nu(\Gamma_{\mathbf{a}})$.

Let M be a matching of $\Gamma_{\mathbf{a}}$ with $|M| = \nu(\Gamma_{\mathbf{a}})$. Consider M as a matching of $\Gamma_{\mathbf{a}+m\mathbf{e}_i}$. By the above mentioned weighted version of Berge's theorem, there is an augmenting walk P of M . Adding the odd edges of P to M and deleting the even edges of P from M we obtain a matching M^* of $\Gamma_{\mathbf{a}+m\mathbf{e}_i}$. It is clear that M^* has $\nu(\Gamma_{\mathbf{a}}) + 1$ edges. Hence M^* is not a matching of $\Gamma_{\mathbf{a}}$. This has the following consequences on the vertex i . First, since $\Gamma_{\mathbf{a}}$ differs from $\Gamma_{\mathbf{a}+m\mathbf{e}_i}$ only by the weight of i , i must appear in P and $w_i^* > a_i$, where w_j^* denote the appearing times of a vertex j in M^* . Let i_f and i_l denote the first and the last vertex of P . Then $w_j^* = w_j \leq a_j$ if $j \neq i_f, i_l$. Therefore, $i = i_f$ or $i = i_l$. Without restriction we may assume that $i = i_f$. If $i \neq i_l$, then $w_i^* = w_i + 1 \leq a_i + 1$. Together with the condition $w_i^* > a_i$, this implies $w_i = a_i$. Thus, if $w_i < a_i$, we must have $i = i_f = i_l$. In this case, $w_i^* = w_i + 2 \leq a_i + 1$. Hence $w_i^* = a_i + 1$, $w_i = a_i - 1$, and $w_j^* = w_j \leq a_j$ for $j \neq i$.

Let W denote the set of the vertices $i \in U$ with $w_i = a_i - 1$. Then $w_i = a_i$ for $i \notin W$. Therefore,

$$\sum_{i \in U} a_i = \sum_{i \in U} w_i + |W| = 2|M| + |W| = 2\nu(\Gamma_{\mathbf{a}}) + |W|.$$

The vertices of W are not adjacent to each other because otherwise we could add any edge of Γ with endpoints in W to M to obtain a matching of $\Gamma_{\mathbf{a}}$ with $\nu(\Gamma_{\mathbf{a}}) + 1$ edges. For $i \in W$ we denote by E_i the first edge of the augmenting walk P . As we have seen above, $i \in E_i$. Let h be the other endpoint of E_i . Then every other vertex $j \in W$ is not adjacent to h because otherwise we could replace E_i by the edge $\{j, h\}$ to obtain from M^* a matching of $\Gamma_{\mathbf{a}}$ with $\nu(\Gamma_{\mathbf{a}}) + 1$ edges. This follows from the facts that $w_i^* = a_i + 1$, $w_j^* = w_j = a_j - 1$. Therefore, $E_i \cap E_j = \emptyset$. From this it follows that the edges E_i , $i \in W$, form a matching of Γ . Hence $|W| \leq \nu(\Gamma) \leq \nu(\Gamma_{\mathbf{a}})$. So we obtain $\sum_{i \in U} a_i \leq 3\nu(\Gamma_{\mathbf{a}}) \leq 3(t-1)$. \square

Remark. The bound $\sum_{i \in U} a_i \leq 3(t-1)$ is sharp as can be seen from the case Ω is the simple graph of $t-1$ disconnected triangles.

By Theorem 2.1, we immediately obtain the following interesting consequence.

Corollary 2.7. *If $x^{\mathbf{a}} \in \tilde{I}^t \setminus I^t$, then $\deg x^{\mathbf{a}} \leq 3(t-1)$.*

Lemma 2.8. *The base graph of any t -saturating weighted graph contains at least one odd cycle of length $\leq 2t-1$.*

Proof. Let Γ be the base graph of a t -saturating weighted graph Ω . Then there is $\mathbf{a} \in \mathbb{N}^n$ such that $\Omega = \Gamma_{\mathbf{a}}$. Since $V = V_{\mathbf{a}}$, $x^{\mathbf{a}} \in \tilde{I}^t \setminus I^t$ by Theorem 2.1. Hence $\tilde{I}^t \neq I^t$. Let $I^{(t)}$ denotes the t -th symbolic power of I , which is the intersection of the primary

components of the minimal associated primes of I^t . Then $I^{(t)} \supseteq \tilde{I}^t$. Hence $I^{(t)} \neq I^t$. By [10, Lemma 3.10] (see also [12, Lemma 5.8, Theorem 5.9]), this implies that Γ has at least one odd cycle of length $\leq 2t - 1$. \square

Remark. A 2-saturating weighted graph must be the graph of a triangle because it contains a triangle by Lemma 2.8 and any other weighted graph containing a triangle has matching number at least 2.

By definition, if Ω is a t -saturating weighted graph, then Ω is s -saturating for $s = \nu(\Omega) + 1$. This property is preserved by the connected components of Ω in the following sense.

Lemma 2.9. *Let $\Omega_1, \dots, \Omega_m$ be the connected components of a weighted graph Ω . Let $t = \nu(\Omega) + 1$ and $t_i = \nu(\Omega_i) + 1$, $i = 1, \dots, m$. Then Ω is t -saturating if and only if Ω_i is t_i -saturating for $i = 1, \dots, m$.*

Proof. It is easy to see that $\nu(\Omega) = \sum_{i=1}^m \nu(\Omega_i)$. Hence $t - 1 = \sum_{i=1}^m (t_i - 1)$. Assume that Ω is t -saturating. Then $\deg_{\Omega}(i) + \nu(\Omega - N_{\Omega}(i)) \geq t$ for every vertex i of Ω . If i is a vertex of Ω_1 , we have $N_{\Omega}(i) = N_{\Omega_1}(i)$. Hence $\deg_{\Omega_1}(i) = \deg_{\Omega}(i)$. Moreover, the connected components of $\Omega - N_{\Omega}(i)$ are $\Omega_1 - N_{\Omega_1}(i)$ and $\Omega_2, \dots, \Omega_m$. Therefore,

$$\nu(\Omega - N_{\Omega}(i)) = \nu(\Omega_1 - N_{\Omega_1}(i)) + \sum_{j=2}^m \nu(\Omega_j) = \nu(\Omega_1 - N_{\Omega_1}(i)) + \sum_{j=2}^m (t_j - 1).$$

From this it follows that

$$\deg_{\Omega_1}(i) + \nu(\Omega_1 - N_{\Omega_1}(i)) = \deg_{\Omega}(i) + \nu(\Omega - N_{\Omega}(i)) - \sum_{j=2}^m (t_j - 1) \geq t - \sum_{j=2}^m (t_j - 1) = t_1.$$

Hence Ω_1 is t_1 -saturating. Similarly, Ω_i is t_i -saturating for $i = 2, \dots, m$ as well.

Conversely, assume that Ω_i is t_i -saturating for $i = 1, \dots, m$. Let i be an arbitrary vertex of Ω . Without loss of generality we may assume that i is a vertex of Ω_1 . Then $\deg_{\Omega_1}(i) + \nu(\Omega_1 - N_{\Omega_1}(i)) \geq t_1$. Therefore,

$$\deg_{\Omega}(i) + \nu(\Omega - N_{\Omega}(i)) = \deg_{\Omega_1}(i) + \nu(\Omega_1 - N_{\Omega_1}(i)) + \sum_{j=2}^m (t_j - 1) \geq t_1 + \sum_{j=2}^m (t_j - 1) = t.$$

Hence Ω is t -saturating, as required. \square

Using the above properties of t -saturating weighted graphs we obtain the following necessary condition for P_F to be an embedded associated prime of I^t .

Theorem 2.10. *Let P_F be an arbitrary embedded associated prime of I^t . Then F is minimal among the covers of Γ containing $N[U]$ for a subset U of V such that every connected component of the induced graph Γ_U contains at least one odd cycle of length $\leq 2t - 1$.*

Proof. By Theorem 2.4, an embedded associated prime of I^t is of the form P_F , where F is minimal among the covers of Γ containing $N[V_{\mathbf{a}}]$ for a t -saturating weighted graph $\Gamma_{\mathbf{a}}$. By Lemma 2.8 and Lemma 2.9, every connected component of the induced graph $\Gamma_{V_{\mathbf{a}}}$ must contain at least one odd cycle of length $\leq 2t - 1$. \square

It is easy to find examples showing that the above condition is not sufficient for P_F to be an embedded associated prime of I^t .

Example 2.11. Let Γ be the graph of the edges $\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}$ and $U = \{1, 2, 3, 4\}$. Then $N[U] = V = \{1, 2, 3, 4, 5\}$. Hence V is the only cover containing $N[U]$. However, $P_V = \mathfrak{m}$ is not an associated prime of I^2 . For, if \mathfrak{m} is an associated prime of I^2 , then by Theorem 2.4, V is minimal among the covers containing $N[V_{\mathbf{a}}]$ for some $\mathbf{a} \in \mathbb{N}^5$ such that $\Gamma_{\mathbf{a}}$ is 2-saturating. This condition implies that $\Gamma_{\mathbf{a}}$ is a triangle. Hence $V_{\mathbf{a}} = \{1, 2, 3\}$. However, V is not a minimal cover containing $N[V_{\mathbf{a}}] = \{1, 2, 3, 4\}$.

3. STRONGLY SATURATING WEIGHTED GRAPHS

Let I be the edge ideal of a simple graph Γ as in the previous section. The aim of this section is to introduce a special class of weighted graphs whose existence on Γ gives rise to embedded associated primes of I^t , $t \geq 2$. Note that the existence of a t -saturating weighted graph on Γ alone does not necessarily leads to an embedded associated prime of I^t . By Theorem 2.4 we need a further condition on vertices outside such a weighted graph.

We call a weighted graph Ω *strongly t -saturating* if $\nu(\Omega) < t$ and $\nu(\Omega - j) \geq t - a_j$ for all vertices j of Ω , where a_j is the weight of j . There many strongly t -saturating graphs (whose vertices have weight 1). For instance, it is easy to check that the union of $t - 1$ triangles which are disconnected or which meet each other at a common vertex is strongly t -saturating. Another examples are graphs on $2t - 1$ vertices which contain an odd cycle of length $2t - 1$.

Recall that Ω is t -saturating if $\nu(\Omega) < t$ and $\nu(\Omega - N_{\Omega}(i)) \geq t - \sum_{j \in N_{\Omega}(i)} a_j$ for all vertices i of Ω . Then a strongly t -saturating weighted graph is t -saturating by the following property.

Lemma 3.1. *Let Ω be a strongly t -saturating weighted graph Ω . Let N be a non-empty set of vertices of Ω . Then $\nu(\Omega - N) \geq t - \sum_{j \in N} a_j$.*

Proof. Choose a vertex $i \in N$. Let Ω' be the weighted subgraph of $\Omega - i$ whose edges contain at least one vertex in $N \setminus i$ and whose vertices have the same weight as in Ω . Then the number of edges of a matching of Ω' can not exceed the appearing times of the vertices of $N \setminus i$ in these edges. Hence $\nu(\Omega') \leq \sum_{j \in N} a_j - a_i$. Since every matching of $\Omega - i$ is the disjoint union of a matching of Ω' and a matching of $\Omega - N$, we have

$$\nu(\Omega - i) \leq \nu(\Omega') + \nu(\Omega - N) \leq \sum_{j \in N} a_j - a_i + \nu(\Omega - N).$$

This implies $\nu(\Omega - N) \geq \nu(\Omega - i) + a_i - \sum_{j \in N} a_j$. Since Ω is strongly t -saturating, $\nu(\Omega - i) \geq t - a_i$. Therefore, $\nu(\Omega - N) \geq t - \sum_{j \in N} a_j$. \square

By definition, if Ω is a strongly t -saturating weighted graph, then Ω is strongly s -saturating for $s = \nu(\Omega) + 1$. This property is preserved by the connected components of Ω as follows.

Lemma 3.2. *Let $\Omega_1, \dots, \Omega_m$ be the weighted connected components of a weighted graph Ω . Let $t = \nu(\Omega) + 1$ and $t_i = \nu(\Omega_i) + 1$, $i = 1, \dots, m$. Then Ω is strongly t -saturating if and only if Ω_i is strongly t_i -saturating for $i = 1, \dots, m$.*

Proof. The proof is similar to that of Lemma 2.9. Hence we omit it. \square

The following sufficient condition for an embedded associated prime of I^t shows that if there exists a strongly t -saturating weighted graph on a subset of V , then we can construct an embedded prime of I^t .

Theorem 3.3. *Let F be a cover of Γ which is minimal among the covers containing $N[U]$ for a set $U \subseteq V$ such that there exists a strongly t -saturating weighted graph on U . Then P_F is an embedded associated prime of I^t .*

Proof. Let $\mathbf{a} \in \mathbb{N}^n$ such that $V_{\mathbf{a}} = U$ and $\Gamma_{\mathbf{a}}$ is a strongly t -saturating weighted graph. By Theorem 2.4 it suffices to show that $\nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(i)) \geq t - \deg_{\mathbf{a}}(i)$ for $i \in \text{core}(F) \setminus V_{\mathbf{a}}$. As in the proof of Theorem 2.4, the assumption on F implies $\text{core}(F) \subseteq N[V_{\mathbf{a}}]$. Hence i is adjacent to at least one vertex $j \in V_{\mathbf{a}}$. This means $N_{\mathbf{a}}(i) \neq \emptyset$. Since $\deg_{\mathbf{a}}(i) = \sum_{j \in N_{\mathbf{a}}(i)} a_j$, the above inequality follows from Lemma 3.1. \square

We don't know whether the above sufficient condition is also necessary.

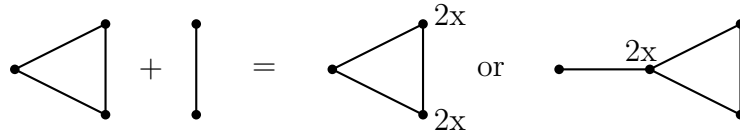
In the following we are interested in strongly s -saturating subgraphs of Γ on $2s - 1$ vertices for some $s < t$. The reason is that we can add edges to such a subgraph to obtain strongly t -saturating weighted graphs. The idea originates from a construction of Chen, Morey and Sung in [2, Theorem 3.7].

Lemma 3.4. *Let U be a subset of V such that Γ_U is connected and contains a strongly s -saturating graph on $2s - 1$ vertices. Then there exists a strongly t -saturating weighted graph Ω on U with $\nu(\Omega) = t - 1$ for all $t \geq |U| - s + 1$.*

Proof. We can reformulate the assertion as there exists $\mathbf{a} \in \mathbb{N}^n$ with $V_{\mathbf{a}} = U$ such that $\Gamma_{\mathbf{a}}$ is a strongly t -saturating weighted graph with $\nu(\Gamma_{\mathbf{a}}) = t - 1$ for all $t \geq |U| - s + 1$. By the assumption, there exists a strongly s -saturating graph C on a set $W \subseteq U$ of $2s - 1$ vertices. Since $s > \nu(C) \geq \nu(C - i) \geq s - 1$, we have $\nu(C) = \nu(C - i) = s - 1$ for all $i \in W$. Since C is a subgraph of the induced graph Γ_W , $\nu(\Gamma_W) \geq \nu(C) = s - 1$. On the other hand, $\nu(\Gamma_W) \leq s - 1$ because Γ_W has $2s - 1$ vertices. Therefore, $\nu(\Gamma_W) = s - 1$. Since $\nu(\Gamma_W) \geq \nu(\Gamma_W - i) \geq \nu(C - i) = s - 1$, we also have $\nu(\Gamma_W - i) = s - 1$ for all $i \in W$. Therefore, Γ_W is a strongly s -saturating graph. Let $\mathbf{c} \in \mathbb{N}^n$ such that $\Gamma_{\mathbf{c}} = \Gamma_W$. Starting from $\Gamma_{\mathbf{c}}$ we can build up a weighted graph $\Gamma_{\mathbf{a}}$ as in the assertion by using the following claim.

Claim. Let $\mathbf{b} \in \mathbb{N}^n$ such that $\Gamma_{\mathbf{b}}$ is a strongly t -saturating weighted graph with $\sum_{i=1}^n b_i = 2t - 1$ and $\nu(\Gamma_{\mathbf{b}}) = \nu(\Gamma_{\mathbf{b} - \mathbf{e}_i}) = t - 1$ for all $i \in V_{\mathbf{b}}$. Let $\{h, j\}$ be an edge of Γ such that $h \in V_{\mathbf{b}}$. Put $\mathbf{a} = \mathbf{b} + \mathbf{e}_h + \mathbf{e}_j$. Then $\Gamma_{\mathbf{a}}$ is a strongly $(t + 1)$ -saturating weighted graph with $\sum_{i=1}^n a_i = 2t + 1$ and $\nu(\Gamma_{\mathbf{a}}) = \nu(\Gamma_{\mathbf{a} - \mathbf{e}_i}) = t$ for all $i \in V_{\mathbf{a}}$.

For convenience we say that $\Gamma_{\mathbf{a}}$ is obtained from $\Gamma_{\mathbf{b}}$ by adding an edge.



Proof of the claim. It is clear that $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i + 2 = 2t + 1$ and $\nu(\Gamma_{\mathbf{a}}) \geq \nu(\Gamma_{\mathbf{b}}) + 1 = t$. Since $2\nu(\Gamma_{\mathbf{a}}) \leq \sum_{i=1}^n a_i$, we must have $\nu(\Gamma_{\mathbf{a}}) = t$.

To show $\nu(\Gamma_{\mathbf{a} - \mathbf{e}_i}) = t$ for all $i \in V_{\mathbf{a}}$ we note that $\nu(\Gamma_{\mathbf{a} - \mathbf{e}_i}) \leq \nu(\Gamma_{\mathbf{a}}) = t$. If $i \in V_{\mathbf{b}}$, $\Gamma_{\mathbf{a} - \mathbf{e}_i} = \Gamma_{\mathbf{b} - \mathbf{e}_i + \mathbf{e}_h + \mathbf{e}_j}$. Hence $\nu(\Gamma_{\mathbf{a} - \mathbf{e}_i}) \geq \nu(\Gamma_{\mathbf{b} - \mathbf{e}_i}) + 1 = t$. If $i \notin V_{\mathbf{b}}$, then $i = j$ because

$V_{\mathbf{a}} = V_{\mathbf{b}} \cup \{j\}$. Hence $\Gamma_{\mathbf{a}-e_i} = \Gamma_{\mathbf{b}+e_h}$. Let v be a vertex of $V_{\mathbf{b}}$ adjacent to h . Then $\Gamma_{\mathbf{b}+e_h} = \Gamma_{\mathbf{b}-e_v+e_h+e_v}$. Hence $\nu(\Gamma_{\mathbf{a}-e_i}) \geq \nu(\Gamma_{\mathbf{b}-e_v}) + 1 = t$. Thus, $\nu(\Gamma_{\mathbf{a}-e_i}) = t$ for all $i \in V_{\mathbf{a}}$.

It remains to show that $\nu(\Gamma_{\mathbf{a}} - i) \geq t + 1 - a_i$ for all $i \in V_{\mathbf{a}}$. If $i \in V_{\mathbf{b}}$, we have $\nu(\Gamma_{\mathbf{b}} - i) \geq t - b_i$. For $i = h$ or $i = j$, we have $a_i = b_i + 1$. Hence $\nu(\Gamma_{\mathbf{a}} - i) \geq \nu(\Gamma_{\mathbf{b}} - i) \geq t + 1 - a_i$. For $i \neq j, h$, we have $a_i = b_i$. Hence $\nu(\Gamma_{\mathbf{a}} - i) \geq \nu(\Gamma_{\mathbf{b}} - i) + 1 \geq t + 1 - a_i$. If $i \notin V_{\mathbf{b}}$, then $a_i = 1$ and $\Gamma_{\mathbf{a}} - i = \Gamma_{\mathbf{a}-e_i}$. As we have seen above, $\nu(\Gamma_{\mathbf{a}-e_i}) = t$. Hence $\nu(\Gamma_{\mathbf{a}} - i) = t + 1 - a_i$. The proof of the claim is now complete.

Now we continue with the proof of Lemma 3.4. Since Γ_U is connected, we can use the claim successively to add $|U| - 2s + 1$ edges of Γ_U to $\Gamma_{\mathbf{c}}$ to obtain a strongly weighted graph $\Gamma_{\mathbf{a}}$ with $V_{\mathbf{a}} = U$ and $\nu(\Gamma_{\mathbf{a}}) = t - 1$ for $t = s + (|U| - 2s + 1) = |U| - s + 1$. For $t > |U| - s + 1$ we only need to add more edges of Γ_U to get $\Gamma_{\mathbf{a}}$. \square

Remark. Not all strongly t -saturating weighted graphs $\Gamma_{\mathbf{a}}$ with $\sum_{i=1}^n a_i = 2t + 1$ and $\nu(\Gamma_{\mathbf{a}}) = \nu(\Gamma_{\mathbf{a}-e_i}) = t$ for all $i \in V_{\mathbf{a}}$ can be obtained from a strongly s -saturating graph on $2s - 1$ vertices, $s < t$, by adding edges. The union of a triangle and a rectangle meeting only at a vertex of weight 2 is such a weighted graph.

By Theorem 2.10, if P_F is an embedded associated prime of some power of I , then F is minimal among the covers of Γ containing $N[U]$ for a subset U of V such that every connected component of Γ_U contains at least one odd cycle. An odd cycle of length $2s - 1$ is obviously a strongly s -saturating graph on $2s - 1$ vertices. Therefore, using Theorem 3.3 and Lemma 3.4 we can say for which t is P_F an associated primes of t .

Theorem 3.5. *Let F be a cover of Γ which is minimal among the covers containing $N[U]$ for a set $U \subseteq V$ such that every connected components of Γ_U contains at least one odd cycle. Assume that Γ_U has m connected components and let s_i be the largest number such that the i -th component of Γ_U in some order contains a strongly s_i -saturating graph on $2s_i - 1$ vertices, $i = 1, \dots, m$. Then P_F is an embedded associated prime of I^t for all $t \geq |U| - \sum_{i=1}^m s_i + 1$.*

Proof. Let Γ_i be the i -th connected component of Γ_U and U_i its vertex set, $i = 1, \dots, m$. By Lemma 3.4, there exists a strongly t_i -saturating weighted graph Ω_i on U_i with $\nu(\Omega_i) = t_i - 1$ for $t_i \geq |U_i| - s_i + 1$. Let Ω be the weighted graph whose connected components are $\Omega_1, \dots, \Omega_m$. Then $\nu(\Omega) = \sum_{i=1}^m \nu(\Omega_i) = \sum_{i=1}^m (t_i - 1)$. By Lemma 3.2, Ω is a strongly t -saturating graph for $t = \sum_{i=1}^m (t_i - 1) + 1$. Now we can apply Theorem 3.3 to deduce that P_F is an associated prime of I^t for $t \geq \sum_{i=1}^m (|U_i| - s_i) + 1 = |U| - \sum_{i=1}^m s_i + 1$. \square

Let $\text{Ass}(I^t)$ denote the set of the associated primes of I^t . By a result of Brodmann [1], there exists a number t_0 such that $\text{Ass}(I^t) = \text{Ass}(I^{t+1})$ for $t \geq t_0$. The stable set $\text{Ass}(I^{t_0})$ is denoted by $\text{Ass}^\infty(I)$. Note that $\text{Ass}(I^t) \subseteq \text{Ass}(I^{t+1})$ for all $t \geq 1$ by Martinez-Bernal, Morey and Villarreal [9, Theorem 2.15]. In [2, Theorem 4.1] Chen, Morey and Sung gave a combinatorial characterization of $\text{Ass}^\infty(I)$ for the case Γ is a connected graph. However, their description of $\text{Ass}^\infty(I)$ is recursive and too complicated to be recalled here. As an immediate consequence of Theorem 2.10 and Theorem 3.5 we obtain the following simple characterization of $\text{Ass}^\infty(I)$.

Corollary 3.6. *Let F be a cover of Γ . Then P_F belongs to $\text{Ass}^\infty(I)$ if and only if F is a minimal cover or minimal among the covers of Γ containing $N[U]$ for a subset U of V .*

V such that every connected component of the induced graph Γ_U contains at least one odd cycle.

We can also give a good upper bound for the smallest number t_0 with the property $\text{Ass}(I^t) = \text{Ass}(I^{t+1})$ for $t \geq t_0$. This number is called the *index of stability* of $\text{Ass}(I^t)$ and denoted by $\text{astab}(I)$ [7]. Note that $\text{astab}(I) = 1$ if Γ is a bipartite graph by a result of Simis, Vasconcelos and Villarreal [12, Theorem 5.9].

Let U be a subset of V such that each connected component of Γ_U contains at least one odd cycle. Set $s(U) = |U| - \sum_{i=1}^m s_i + 1$ if m is the number of connected components of Γ_u and s_i is the largest number such that the i -component (in some order) contains a strongly s_i -saturating graph on $2s_i - 1$ vertices. Let $s(\Gamma)$ denote the maximum of all such $s(U)$, where we set $s(\Gamma) = 1$ if Γ is a bipartite graph.

Corollary 3.7. $\text{astab}(I) \leq s(\Gamma)$.

Proof. Let F be a cover of Γ such that P_F is an associated prime of some of I for $t \gg 0$. Without restriction we may assume that P_F is an embedded associated prime. By Corollary 2.10, F is minimal among the covers of Γ containing $N[U]$ for a set $U \subseteq V$ such that every connected component of the induced graph Γ_U contains at least one odd cycle. By Theorem 3.5, P_F is an associated prime of I^t for all $t \geq s(\Gamma)$. \square

Chen, Morey, and Sung [2, Proposition 4.2] proved that if Γ is connected and non-bipartite, then $\text{astab}(I) \leq n - s$ if $2s - 1$ is the minimal length of odd cycles in Γ and $n > 2s - 1$. The following example shows that the bound of Corollary 3.7 is much better than this bound.

Example 3.8. Let Γ be the union of $t - 1$ triangles meeting each other at a common vertex, $t > 2$. Since Γ has $2t - 1$ vertices, $\text{astab}(I) \leq 2t - 3$ by the bound of Chen, Morey, and Sung. On the other hand, we have $s(\Gamma) = t$, which implies $\text{astab}(I) \leq t$ by Corollary 3.7. To prove $s(\Gamma) = t$ we observe that if U is a subset of V such that Γ_U contains at least one triangle, then Γ_U is connected. Suppose that Γ_U contains $s - 1$ triangles of Γ . Let C denote the union of these triangles. It is easy to see that C is a strongly s -saturating graph on $2s - 1$ vertices. The remaining $|U| - (2s - 1)$ vertices of U must belong to different triangles of Γ outside of C . Hence $|U| - (2s - 1) \leq t - s$. Therefore, $s(U) = |U| - s + 1 \leq t$. In particular, $s(V) = (2t - 1) - t + 1 = t$. So we can conclude that $s(\Gamma) = t$.

4. THE SECOND AND THIRD POWERS OF EDGE IDEALS

In this section we describe the associated primes of I^t for $t = 2, 3$, where I is the edge ideal of an arbitrary simple graph Γ . This will be achieved by classifying the induced subgraphs Γ_U such that the associated primes of I^t correspond to the covers of Γ which are minimal among the covers containing $N[U]$.

The associated primes of I^2 has been described by Herzog and Hibi [6] and Terai and Trung [15]. Their results are immediate consequences of our general approach.

Theorem 4.1. [15, Theorem 3.8] *Let F be a cover of Γ . Then P_F is an associated prime of I^2 if and only if F is a minimal cover or minimal among the covers containing the closed neighborhood of a triangle.*

Proof. We only need to characterize embedded associated primes of I^2 . Note that a 2-saturating weighted graph is a triangle. Then the assertion follows from Theorem 2.4 and Theorem 3.3. \square

Corollary 4.2. [6, Theorem 2.1], [15, Theorem 2.8] $\text{depth } R/I^2 > 0$ if and only if Γ has no dominating triangle.

Proof. Note that $\text{depth } R/I^2 > 0$ if and only if $\mathfrak{m} = P_V$ is not an associated prime of I^2 . Since V is minimal among the covers containing a subset N if and only if $V = N$, we only need to apply Theorem 4.1 to the case $F = V$ to obtain the assertion. \square

For $t = 3$ we first describe the monomials of $\tilde{I}^3 \setminus I^3$. To simplify our arguments we say that a weighted graph Ω is spanned by a weighted subgraph Ω' of Ω or Ω' is a *spanning weighted graph* of Ω if they share the same vertices with the same weights (their edges may be different). Moreover, we say that Ω is a *proper extension* of Ω' if Ω is not spanned by Ω' .

Theorem 4.3. Let $\mathbf{a} \in \mathbb{N}^n$. Then $x^{\mathbf{a}} \in \tilde{I}^3 \setminus I^3$ if and only if $V_{\mathbf{a}}$ is a dominating set of Γ and $\Gamma_{\mathbf{a}}$ satisfies one of the following conditions:

- (i) $\Gamma_{\mathbf{a}}$ is a triangle with weight vector $(2, 2, 1)$,
- (ii) $\Gamma_{\mathbf{a}}$ is spanned by a union of an edge and a triangle meeting at a vertex of weight 2,
- (iii) $\Gamma_{\mathbf{a}}$ is a union of two non-adjacent triangles,
- (iv) $\Gamma_{\mathbf{a}}$ is spanned by the union of two triangles meeting at a vertex,
- (v) $\Gamma_{\mathbf{a}}$ is spanned by a pentagon,
- (vi) $\Gamma_{\mathbf{a}}$ is a complete graph K_4 and every vertex of $V \setminus V_{\mathbf{a}}$ is adjacent to at least two vertices of $V_{\mathbf{a}}$.

Proof. It is easy to check that the weighted graphs $\Gamma_{\mathbf{a}}$ listed above satisfy the conditions of Theorem 2.1 for $t = 3$, which implies $x^{\mathbf{a}} \in \tilde{I}^3 \setminus I^3$.

To prove the converse let $x^{\mathbf{a}} \in \tilde{I}^3 \setminus I^3$. Then $\Gamma_{\mathbf{a}}$ is a 3-saturating weighted graph by Theorem 2.1, and $V_{\mathbf{a}}$ is a dominating set of Γ by Lemma 2.2. By the definition of 3-saturating weighted graph, $\nu(\Gamma_{\mathbf{a}}) \leq 2$. If $\nu(\Gamma_{\mathbf{a}}) = 1$, then $\Gamma_{\mathbf{a}}$ is 2-saturating. A 2-saturating weighted graph must be a triangle. Since a triangle is not 3-saturating, we get a contradiction. Thus, $\nu(\Gamma_{\mathbf{a}}) = 2$.

If $\Gamma_{\mathbf{a}}$ has more than one connected components, then there are only two components and each component must be 2-saturating by Lemma 2.9. Hence $\Gamma_{\mathbf{a}}$ must be a union of two disjoint triangles. That is Case (iii). So we may assume that $\Gamma_{\mathbf{a}}$ is connected.

By Lemma 2.8, $\Gamma_{\mathbf{a}}$ contains at least a cycle or a pentagon. If $\Gamma_{\mathbf{a}}$ contains a pentagon, then $\Gamma_{\mathbf{a}}$ is spanned by this pentagon because any proper extension of a pentagon has matching number ≥ 3 . That is Case (v). So we may further assume that $\Gamma_{\mathbf{a}}$ contains a triangle, say $C = \{i, j, h\}$, and no pentagon. Note that the weight of every vertex of C is at most 2 by Lemma 2.5(i).

If two vertices of C has weight 2, say $a_i = a_j = 2$, then $\Gamma_{\mathbf{a}}$ must be the weighted graph on C with weight vector $(2, 2, 1)$ because any proper extension of this weighted graph has matching number ≥ 3 . That is Case (i).

If only a vertex of C has weight 2, say $a_i = 2$, then $\deg_{\mathbf{a}}(i) > 2$ by Lemma 2.5(i). Since $\deg_{\mathbf{a}}(i) = \sum_{v \in N(i)} a_v$ and $a_j = a_h = 1$, $N(i)$ must contain at least a vertex

$v \neq j, h$. So $\Gamma_{\mathbf{a}}$ contains the union of the edge $\{i, v\}$ and the triangle C with only i having weight 2. Since any proper extension of this weighted graph has matching number ≥ 3 , $\Gamma_{\mathbf{a}}$ is spanned by this weighted graph. That is Case (ii).

If all vertices of C have weight one, then $\Gamma_{\mathbf{a}}$ has at least a vertex $v \notin C$ because $\nu(C) = 1$. Since $\Gamma_{\mathbf{a}}$ is connected, we may assume that v is adjacent to a vertex of C , say i . Since $a_i = 1$, v is not a leaf vertex of $\Gamma_{\mathbf{a}}$ by Lemma 2.5(ii). So v is adjacent to another vertex of $\Gamma_{\mathbf{a}}$. Now we distinguish two cases.

Case 1: v is adjacent to a vertex of $\Gamma_{\mathbf{a}}$ outside of C , say w (see Figure I). Then $a_v = 1$ because otherwise the edges $\{w, v\}, \{v, i\}, \{j, h\}$ would form a matching of $\Gamma_{\mathbf{a}}$, a contradiction to $\nu(\Gamma_{\mathbf{a}}) = 2$. Thus, w is not a leaf vertex of $\Gamma_{\mathbf{a}}$ by Lemma 2.5(ii). On the other hand, w can not be adjacent to any other vertex $u \neq w$ of $\Gamma_{\mathbf{a}}$ outside of C because otherwise the edges $\{u, w\}, \{v, i\}, \{j, h\}$ would form a matching of $\Gamma_{\mathbf{a}}$, a contradiction to $\nu(\Gamma_{\mathbf{a}}) = 2$. So w is adjacent to at least a vertex of C . If w is adjacent to j or h , $\Gamma_{\mathbf{a}}$ would contain a pentagon on the vertices i, j, h, v, w , a contradiction to our assumption. So w is adjacent to i , and $\Gamma_{\mathbf{a}}$ contains a union of two triangles meeting at a vertex. Since any proper extension of this union has matching number ≥ 3 , $\Gamma_{\mathbf{a}}$ is spanned by this union. That is Case (iv).

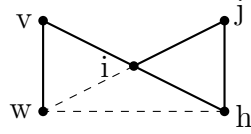


Figure I

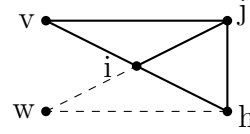


Figure II

Case 2: v is not adjacent to any vertex of $\Gamma_{\mathbf{a}}$ outside of C . Then v is adjacent to another vertex of C , say j (see Figure II). Let w be an arbitrary vertex of $\Gamma_{\mathbf{a}}$ outside of C which is adjacent to a vertex of C . By Case 1, we may assume that w is not adjacent to any vertex of $\Gamma_{\mathbf{a}}$ outside of C . Since $\Gamma_{\mathbf{a}}$ is connected, this implies that $\Gamma_{\mathbf{a}} - C$ consists of only isolated vertices. As a consequence, $\nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(h)) = 0$ because $C \subseteq N_{\mathbf{a}}(h)$. By Theorem 2.1(ii), this implies $\deg_{\mathbf{a}}(h) \geq 3$. Hence $N(h)$ contains at least a vertex of $\Gamma_{\mathbf{a}}$ outside of C . Let w be such a vertex. Like v , the vertex w is adjacent to at least a vertex of C other than h , say i . If $w \neq v$, we would get a pentagon on the vertices i, v, j, h, w , a contradiction to our assumption. So we must have $w = v$. Since v, w are arbitrarily chosen, v must be the unique vertex of $\Gamma_{\mathbf{a}}$ outside of C . Therefore, $\Gamma_{\mathbf{a}}$ is a complete graph K_4 . For every vertex $u \in V \setminus V_{\mathbf{a}}$ we have $\deg_{\mathbf{a}}(u) + \nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(u)) \geq 3$ by Theorem 2.1(ii). Since $V_{\mathbf{a}}$ is a dominating set of Γ , $N_{\mathbf{a}}(u) \neq \emptyset$. Consequently, $\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(u)$ is contained in a triangle. Hence $\nu(\Gamma_{\mathbf{a}} - N_{\mathbf{a}}(u)) \leq 1$. So we get $\deg_{\mathbf{a}}(u) \geq 2$. This means u is adjacent to at least two vertices of $\Gamma_{\mathbf{a}}$. That is Case (vi).

The proof of Theorem 4.3 is now complete. \square

Using Theorem 4.3 we can characterize the associated primes of I^3 as follows.

Theorem 4.4. *Let F be a cover of Γ . Then P_F is an associated prime of I^3 if and only if F is a minimal cover or F is minimal among the covers containing the closed neighborhood of the vertex set of a subgraph of the following forms: a triangle, a union of an edge and a triangle meeting at a vertex, a union of two non-adjacent triangles, a union of two triangles meeting at a vertex, a pentagon.*

Proof. We may assume that F is not a minimal cover of Γ . Then $\text{core}(F) \neq \emptyset$. Let $S := k[x_i \mid i \in \text{core}(F)]$ and $J := I(\Gamma_{\text{core}(F)})$. By Theorem 2.1 and Theorem 2.4, P_F is

an associated prime of I^3 if and only if F is a minimal among the covers containing $N[V_{\mathbf{a}}]$ for some $\mathbf{a} \in \mathbb{N}^n$ such that $x^{\mathbf{a}} \in \widetilde{J^3} \setminus J^3$. By Theorem 4.3, this is the case if and only if the base graph of $\Gamma_{\mathbf{a}}$ is of the above forms. Note that if $\Gamma_{\mathbf{a}}$ is a complete graph K_4 and every vertex of $\text{core}(F) \setminus V_{\mathbf{a}}$ is adjacent to two vertices of $V_{\mathbf{a}}$, then every vertex of $\text{core}(F) \setminus V_{\mathbf{a}}$ is adjacent to every triangle of this K_4 . Hence this case can be passed to the case of a triangle. \square

As in the proof of Corollary 4.2 we obtain from Theorem 4.4 the following criterion.

Corollary 4.5. *depth $R/I^3 > 0$ if and only if Γ has no dominating subgraph of the following forms: a triangle, a union of an edge and a triangle meeting at a vertex, a union of two non-adjacent triangles, a union of two triangles meeting at a vertex, a pentagon.*

REFERENCES

- [1] M. Brodmann, *Asymptotic stability of $\text{Ass}(M/I^n M)$* , Proc. Amer. Math. Soc. **74** (1979), 16–18.
- [2] J. Chen, S. Morey and A. Sung, *The stable set of associated primes of the ideal of a graph*, Rocky Mountain J. Math. **32** (2002), 71–89.
- [3] C.A. Francisco, H.T. Ha and A. Van Tuyl, *Colorings of hypergraphs, perfect graphs, and associated primes of powers of monomial ideals*, J. Algebra **331** (2011), 224–242.
- [4] H.T. Ha and S. Morey, *Embedded associated primes of powers of squarefree monomial ideals*, J. Pure Appl. Algebra **214** (2010), 301–308.
- [5] J. Herzog and T. Hibi, *The depth of powers of an ideal*, J. Algebra **291** (2005), 534–550.
- [6] J. Herzog and T. Hibi, *Bounding the socles of powers of squarefree monomial ideals*, Preprint 2013, arXiv:1308.5400.
- [7] J. Herzog and A. Qureshi, *Persistence and stability properties of powers of ideals*, Preprint 2013, arXiv:1208.4684.
- [8] L. Lovasz and M.D. Plummer, *Matching Theory*, AMS Chelsea Publishing, 2009.
- [9] J. Martinez-Bernal, S. Morey and R. Villarreal, *Associated primes of powers of edge ideals*, Collect. Math. **63** (2012), 361–374.
- [10] G. Rinaldo, N. Terai, and K. Yoshida, *Cohen–Macaulayness for symbolic power ideals of edge ideals*, J. Algebra **347** (2011), 405–430.
- [11] G. Rinaldo, N. Terai, and K. Yoshida, *On the second powers of Stanley–Reisner ideals*, J. Commut. Algebra **3** (2011), 405–430.
- [12] A. Simis, W. Vasconcelos and R. Villarreal, *On the ideal theory of graphs*, J. Algebra **167** (1994), no. 2, 389–416.
- [13] Y. Takayama, *Combinatorial characterizations of generalized Cohen–Macaulay monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **48** (2005), 327–344.
- [14] N. Terai and N.V. Trung, *Cohen–Macaulayness of large powers of Stanley–Reisner ideals*, Adv. in Math. **229** (2012), 711–730.
- [15] N. Terai and N.V. Trung, *On the associated primes and the depth of the second power of squarefree monomial ideals*, J. Pure Appl. Algebra **218** (2014), 1117–1129.
- [16] T.N. Trung, *Stability of depth of power of edge ideals*, Preprint 2013.

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